Fourier Analysis

Dr. Linta Rose

rose.l@incois.gov.in

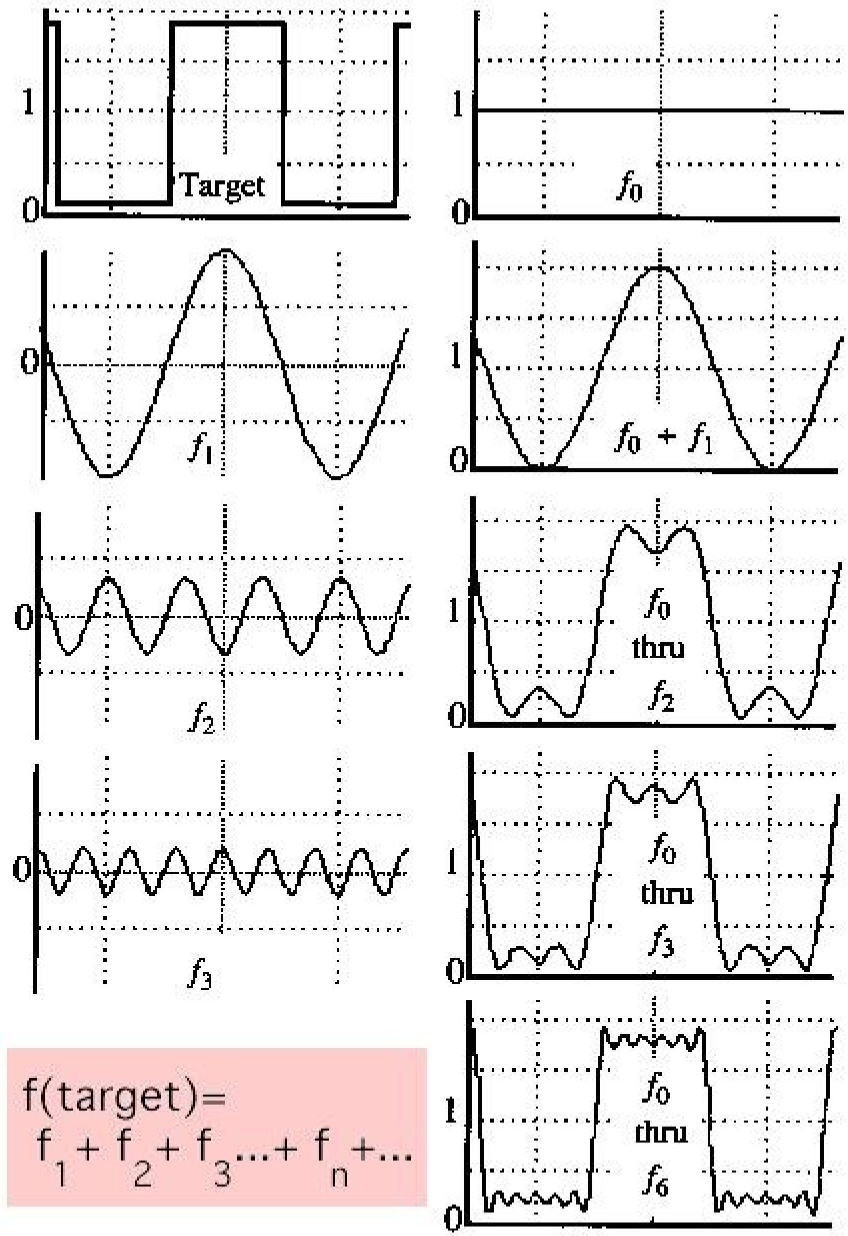
Fourier Series

- Fundamental block:

$$A \sin(\mu x + ¤)$$

- Any continuous waveform can be partitioned into a sum of sinusoidal waves

- Add enough of them to get any signal *f(x)* you want!



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Series

- And if we could add infinite sine waves in that pattern we would **have** a square wave!
- So we can say that:

  a square wave = sin(x) + sin(3x)/3 + sin(5x)/5 + …
- This is the idea of a Fourier series. It simply splits the data into a series of sine waves.
- How did we know to use sin(3x)/3, sin(5x)/5, etc?

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(nx\frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(nx\frac{\pi}{L}\right)$$

Where:

f(x) is the function we want (such as a square wave)

L is **half of the period** of the function

a_0, a_n and b_n are **coefficients** that we need to calculate!

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

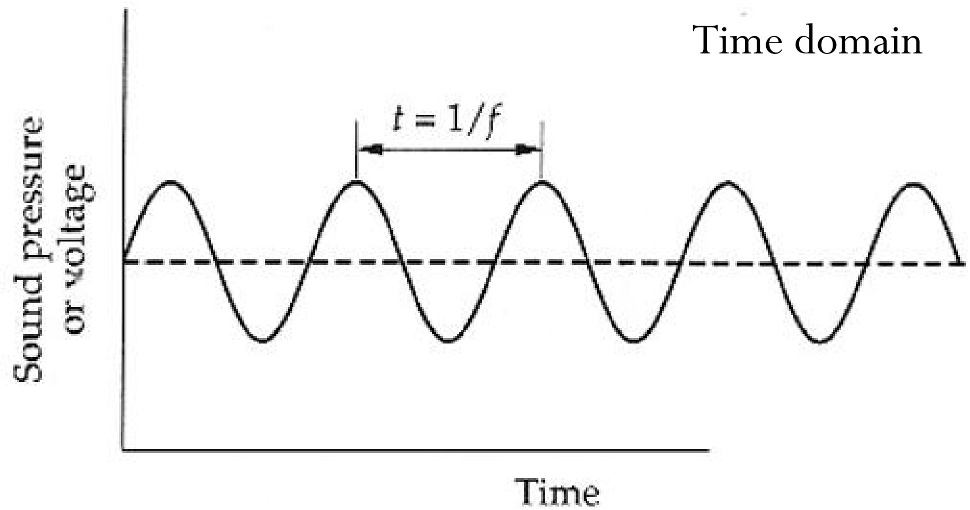
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(nx\frac{\pi}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(nx\frac{\pi}{L}\right) dx$$

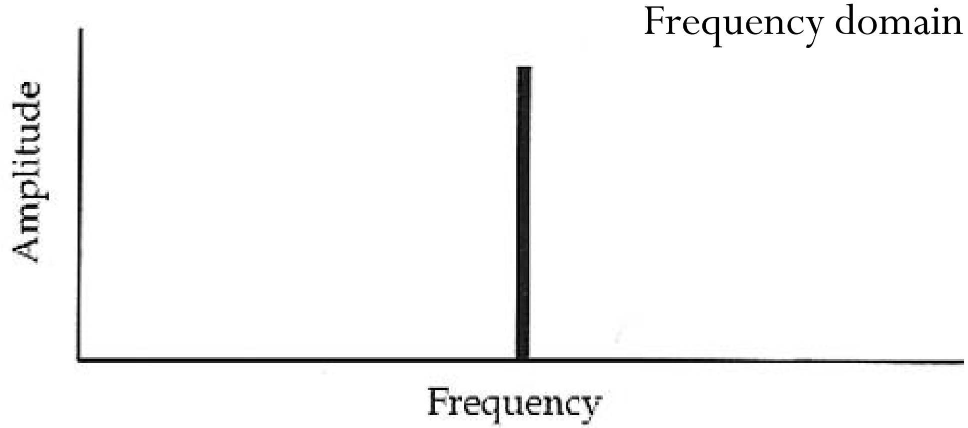
Fourier Analysis

- Commonly used in digital signal processing, image analysis
- To understand the component variability in a time series data
- For oceanographic data, this technique helps in identifying the dominant signals/variability within the time series data for a point location.
- Applied to Time Series data
- Eg. Daily, monthly, seasonal, annual signals and their relative significance
- Any additional processes that affect the data like ENSO, IOD etc, can also be identified as peaks in the Fourier spectrum
- Eg. A time series of SST data will be a combination of several signals, which are approximately sinusoidal, and can estimate the above mentioned processes.

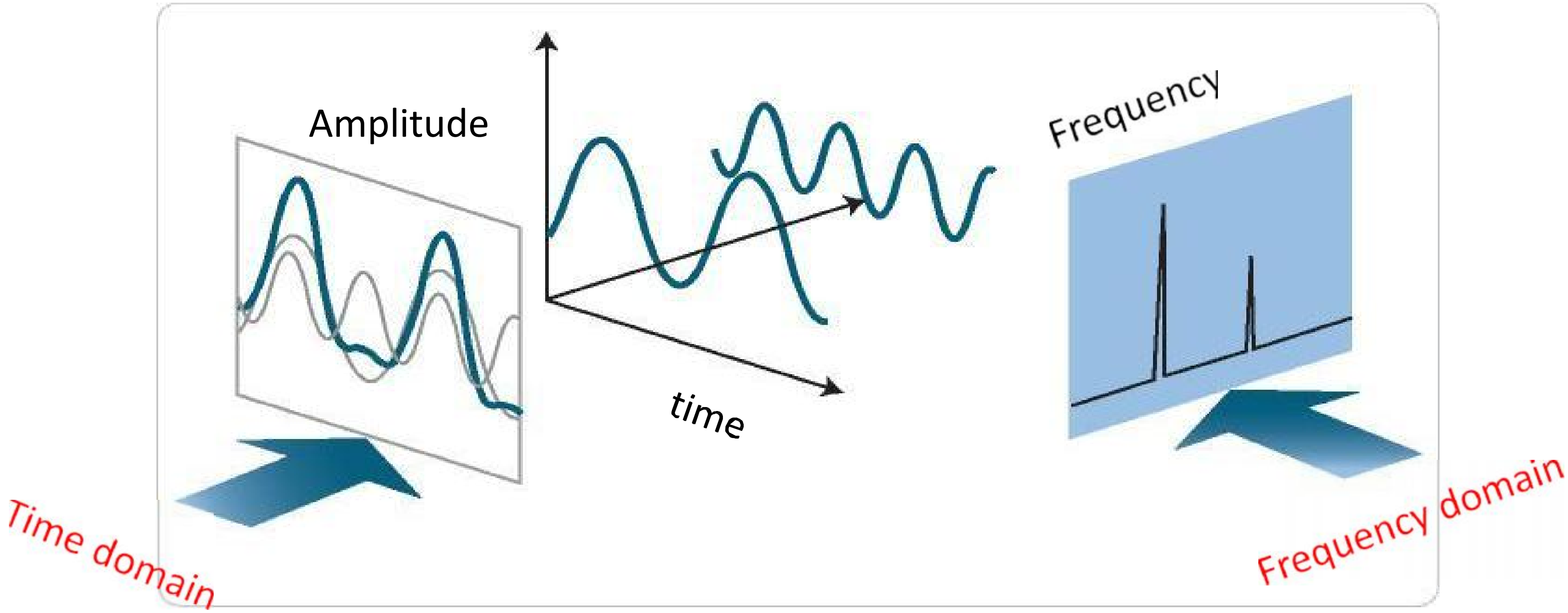
Fourier Transform



Frequency spectrum



Visualizing a Signal – Time Domain & Frequency Domain



- Frequency information is hidden inside the time series data, it is extracted using Fourier Transform
- It splits the time series data into component sine waves

Prism Analogy

White light

Spectrum of colours

Signal $g(t)$ → Fourier Transform → Spectrum $G(\omega)$

Analogy:

a prism which splits white light into a spectrum of colors

white light consists of all frequencies mixed together

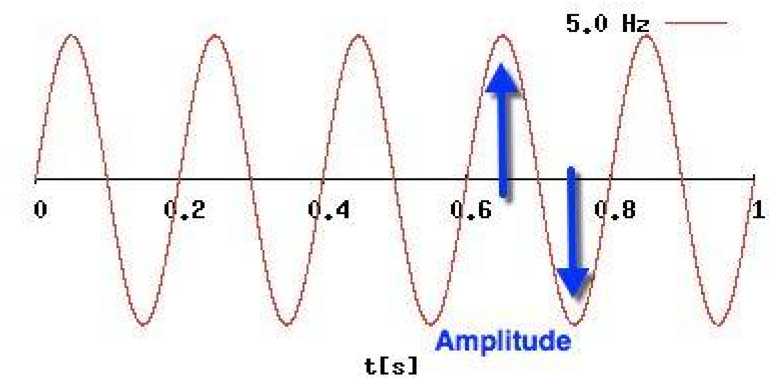
the prism breaks them apart so we can see the separate frequencies

Fourier Transform

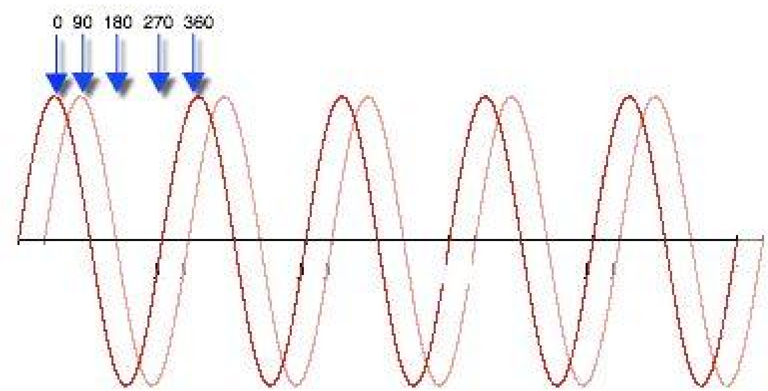
- The amplitude *A* and phase f of the corresponding sine

$$A \sin(\mu x + \text{¤}) \qquad \text{- A simple sine wave}$$

- So our data can be split into a number of such sine waves
- We don't know the amplitude and phase of any of these waves, we need to compute



Amplitude of a wave



Phase of a wave

Terms

- Frequency
- Frequency Spectrum – eg. amplitude/power/energy vs frequency
- Spectrum simply shows the components, here signals that indicate physical processes affecting our data, like diurnal variability
- Discrete Fourier Analysis
- Our data is always discrete.

Fourier Transform

- Our data is *f(x)* or *f(t)*

f(x) → Fourier Transform → ***F(W)***

- We can convert it from the TIME DOMAIN to FREQUENCY DOMAIN using Fourier Transform

F(W) → Inverse Fourier Transform → *f(x)*

  – How can *F* hold both? Complex number trick! $F(\mu) = R(\mu) + iI(\mu)$

$$A = \pm\sqrt{R(\mu)^2 + I(\mu)^2}$$

$$¤ = \tan^{-1}\frac{I(\mu)}{R(\mu)}$$

Definition of Fourier Transform

The Fourier transform (*i.e.*, spectrum) of $f(t)$ is $F(\mu)$:

$$F(\mu) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j\mu t} dt$$

$$f(t) = \mathfrak{F}^{-1}\{F(\mu)\} = \frac{1}{2\text{®}} \int_{-\infty}^{\infty} F(\mu) e^{j\mu t} d\mu$$

Therefore, $f(t) \Leftrightarrow F(\mu)$ is a Fourier Transform pair

Note: Remember $\mu = 2\pi f$

Fourier Transform

- Technique of Fast Fourier Transform (FFT)
- To compute the parameters of the component sine waves- the amplitudes and phases
- The complex algorithm involves powers of 2 for ease of computation
- Zero padding of data to make the data length a power of 2
- Output is a complex number from which amplitude/magnitude and phase of each sine wave can be distinguished.

Terms

- N total number of observations
- Δt the time interval or the sampling interval
- F_s Sampling frequency $f_s = \frac{1}{\Delta t}$

$$\Delta f = \frac{1}{N\Delta t} = \frac{f_s}{N}$$

- Nyquist frequency- the highest frequency that can be detected form the time series data without aliasing
- If our data is monthly signals with time period less than 2 months (eg daily) cannot be identified in the Fourier Transform.

Terms

- Aliasing – signals may get masked/undistinguishable – if we want to understand hourly variation, then we should take samples every half an hour